

Black hole horizons and quantum charged particles

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We point out a structural similarity between the characterization of black hole apparent horizons as stable marginally outer trapped surfaces (MOTS) and the quantum description of a non-relativistic charged particle moving in given magnetic and electric fields on a closed surface. Specifically, the spectral problem of the MOTS-stability operator corresponds to a stationary quantum particle with a formal fine-structure constant α of negative sign. We discuss how such analogy enriches both problems, illustrating this with the insights into the MOTS-spectral problem gained from the analysis of the spectrum of the quantum charged particle Hamiltonian.

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I. INTRODUCTION: A FORMAL ANALOGY

Analogies between physical systems, either of mathematical or physical nature, often play a fundamental catalyst role in conceptual and/or technical developments of the respective theories [1]. We discuss here a mathematical analogy between the descriptions of black hole horizons and quantum charged particles, that opens a domain of cross-fertilization between quantum mechanics and gravitation theory. More specifically, apparent horizons –namely *marginally outer trapped surfaces* (MOTS)– possess a stability notion that guarantees their physical consistency as models of black hole horizons. Such MOTS-stability notion [2] admits a spectral characterization in terms of the so-called *principal* eigenvalue of the operator

$$L_S = -\Delta + 2\Omega^a D_a - \left(|\Omega|^2 - D_a \Omega^a - \frac{1}{2} R_S + G_{ab} k^a \ell^b \right) (1)$$

defined on the apparent horizon \mathcal{S} . The terms modifying the Laplacian Δ on \mathcal{S} are determined by the intrinsic and extrinsic geometry of the apparent horizon and the gravitational equations via the Einstein tensor G_{ab} (see next section for details). The relevant remark in the present context is that under the complexification of the vector Ω^a and the identifications

$$\Omega_a \leftrightarrow \frac{ie}{\hbar c} A_a, \quad R_S \leftrightarrow \frac{4me}{\hbar^2} \phi, \quad G_{ab} k^a \ell^b \leftrightarrow -\frac{2m}{\hbar^2} V \quad (2)$$

the MOTS-stability operator becomes $\frac{\hbar^2}{2m} L_S \leftrightarrow \hat{H}$, where

$$\begin{aligned} \hat{H} = & -\frac{\hbar^2}{2m} \Delta + \frac{i\hbar e}{mc} A^a D_a + \frac{i\hbar e}{2mc} D_a A^a + \frac{e^2}{2mc^2} A_a A^a \\ & + e\phi + V = \frac{1}{2m} \left(-i\hbar D - \frac{e}{c} A \right)^2 + e\phi + V, \end{aligned} \quad (3)$$

is the Hamiltonian of a non-relativistic particle with mass m and charge e moving on \mathcal{S} under magnetic and electric fields with vector and scalar potentials given by A^a and ϕ , and an external potential V . This formal mathematical analogy relies on a simple but crucial remark: the derivative and Ω^a terms in L_S can be collected in a perfect square as follows

$$L_S \psi = \left[- (D - \Omega)^2 + \frac{1}{2} R_S - G_{ab} k^a \ell^b \right] \psi. \quad (4)$$

Beyond its aesthetic appeal, and in spite of the key difference in the self-adjoint nature of the operators, this analogy has the

potential to open bridges between the well-studied quantum particle problem and the rich but largely uncharted MOTS subject, with applications ranging from the MOTS-spectral problem to the spinorial formulation of MOTS stability. We focus here on the study of the full L_S spectrum, a challenging problem formulated in [3] in the setting of a black hole/fluid analogy, but with a definite geometric interest on its own.

II. GEOMETRY AND STABILITY OF MOTS

A. Some MOTS geometry

Let us consider a codimension-2 surface \mathcal{S} , spacelike, closed (compact and without boundary) and orientable, embedded in a n -dimensional spacetime (M, g_{ab}) . The spacetime Levi-Civita connection is denoted by ∇_a with Einstein curvature tensor G_{ab} . Let us denote the induced metric on \mathcal{S} by q_{ab} , with Levi-Civita connection D_a , Ricci scalar R_S , volume form ϵ_{ab} and measure η_S . Let us consider future-oriented null vectors ℓ^a and k^a spanning the normal bundle $T^\perp \mathcal{S}$ and normalized as $\ell^a k_a = -1$. This normalization leaves a null-vector-rescaling freedom by a positive function $f > 0$ respecting time orientation, corresponding to a boost transformation

$$\ell'^a = f \ell^a, \quad k'^a = f^{-1} k^a. \quad (5)$$

We define the expansion $\theta^{(\ell)}$ and Hájiček or rotation form Ω_a

$$\theta^{(\ell)} \equiv q^{ab} \nabla_a \ell_b, \quad \Omega_a \equiv -k^c q^d_a \nabla_d \ell_c, \quad (6)$$

associated with ℓ^a . Considering $v^a = \alpha \ell^a + \beta k^a$, we can write $q^c_a \nabla_c v_b = D_a^\perp v_b + \Theta_{ab}^{(v)}$, with $\Theta_{ab}^{(v)} \equiv q^c_a q^d_b \nabla_c v_d$ and

$$D_a^\perp v_b = (D_a \alpha + \Omega_a \alpha) \ell_b + (D_a \beta - \Omega_a \beta) k_b. \quad (7)$$

The Hájiček form therefore provides a connection on the normal bundle $T^\perp \mathcal{S}$ for the tangent derivative of normal vectors. From a physical perspective it represents a sort of angular momentum density. Given an axial Killing vector ϕ^a on \mathcal{S} ,

$$J[\phi] = \frac{1}{8\pi} \int_{\mathcal{S}} \Omega_a \phi^a \eta_S \quad (8)$$

is the (Komar) angular momentum associated with \mathcal{S} . Regarding the expansion, the surface \mathcal{S} is a marginally outer trapped surface (MOTS) if it satisfies the condition: $\theta^{(\ell)} = 0$. We refer then to ℓ^a as *outgoing* and to k^a as *ingoing* null vectors.

B. MOTS stability and MOTS-stability operator

A MOTS \mathcal{S} is said to be *stable* (more properly, stably outermost [2, 4]) in the ingoing k^a direction if it can be infinitesimally deformed along k^a into a properly (outer) trapped surface \mathcal{S}' , i.e. with $\theta^{(\ell)}|_{\mathcal{S}'} < 0$. Using the deformation operator δ_v along a normal vector v^a (discussed in [2]), this amounts to the existence of a function $\psi > 0$ such that $\delta_{\psi k} \theta^{(\ell)} < 0$. Such a condition admits a spectral characterization in terms of the elliptic operator $L_{\mathcal{S}}$ defined as $L_{\mathcal{S}}\psi \equiv \delta_{\psi(-k)} \theta^{(\ell)}$, with explicit expression (1). The operator $L_{\mathcal{S}}$, namely the MOTS-stability operator, is generically non-selfadjoint [in $L^2(\mathcal{S}, \eta_{\mathcal{S}})$] due to the $2\Omega^a D_a$ term. Therefore, in the eigenvalue problem

$$L_{\mathcal{S}}\psi = \lambda\psi, \quad (9)$$

the λ 's are generically complex. Their real part is bounded below, leading to the definition of the *principal eigenvalue* λ_o of $L_{\mathcal{S}}$ as that with smallest real part. Lemmas 1 and 2 in [2] state that: i) λ_o is real, and ii) \mathcal{S} is stably outermost iff $\lambda_o \geq 0$.

C. MOTS-gauge symmetry

The MOTS geometry described above does not depend on the choice of null normals (subject to $\ell^a k_a = -1$). In this sense, the null vector rescaling freedom (5) is a gauge transformation of the MOTS geometry. We consider now the transformation of the main objects on \mathcal{S} under the rescaling (5).

Lemma 1 (MOTS-gauge transformations). *Under the null normal rescaling $\ell'^a = f\ell^a$, $k'^a = f^{-1}k^a$, with $f > 0$:*

i) *The expansion and Hájíček form transform as*

$$\theta^{(\ell')} = f\theta^{(\ell)}, \quad \Omega'_a = \Omega_a + D_a(\ln f). \quad (10)$$

ii) *The MOTS-stability operator transforms covariantly*

$$(L_{\mathcal{S}})'\psi = fL_{\mathcal{S}}(f^{-1}\psi), \quad (11)$$

where $(L_{\mathcal{S}})'\psi \equiv \delta_{\psi(-k')}\theta^{(\ell')}$.

iii) *The MOTS-eigenvalue problem is invariant under the additional eigenfunction transformation, $\psi' = f\psi$*

$$L_{\mathcal{S}}\psi = \lambda\psi \rightarrow (L_{\mathcal{S}})'\psi' = \lambda\psi'. \quad (12)$$

Proof: Point i) follows directly by plugging (5) into (6). Regarding point ii), although it can be obtained by straightforward substitution of (10) into (1), it is simpler to use the definition of $L_{\mathcal{S}}$. Considering its action on a function ψ

$$\begin{aligned} (L_{\mathcal{S}})'\psi &= \delta_{\psi(-k')}\theta^{(\ell')} = \delta_{\psi(-k')}(f\theta^{(\ell)}) \\ &= \delta_{\psi(-k')}(f)\theta^{(\ell)} + f\delta_{\psi(-k')}\theta^{(\ell)} = f\delta_{\psi(-k')}\theta^{(\ell)} \\ &= f\delta_{\psi(-f^{-1}k)}(\theta^{(\ell)}) = f\delta_{(f^{-1}\psi)(-k)}\theta^{(\ell)} = fL_{\mathcal{S}}(f^{-1}\psi) \end{aligned} \quad (13)$$

where in the first line we have used the $\theta^{(\ell)}$ transformation in (10), the second line uses the Leibnitz rule holding for δ_v and

the MOTS condition, and in the third line we used again the definition of $L_{\mathcal{S}}$. Finally, point iii) follows directly

$$(L_{\mathcal{S}})'\psi' = fL_{\mathcal{S}}(f^{-1}\psi') = fL_{\mathcal{S}}(\psi) = f\lambda\psi = \lambda\psi'. \quad (14)$$

Point i) just states the invariance of the MOTS notion under (5) and the transformation of Ω_a as a connection under the (multiplicative) abelian gauge group \mathbb{R}^+ of positive null rescalings. The latter is consistent with the nature of Ω_a in (7) as a connection in the normal bundle. Point ii), stating the good (covariant) transformation properties of $L_{\mathcal{S}}$ under \mathbb{R}^+ , is the analogue in the present setting of Proposition 4 in [5] concerning the free choice of section of stationary black hole horizons. Point iii) guarantees that the MOTS-eigenvalue problem is well-defined and provides the gauge transformation rule for the associated eigenfunctions. Of course, all these points evoke familiar features of the quantum charged particle.

III. MOTS AND QUANTUM CHARGED PARTICLES

To take a step further from the formal correspondance (2) into a more precise statement, let us review the stationary quantum charged particle (QCP) problem. The Schrödinger equation for a non-relativistic (spin-0) charged particle moving in electromagnetic fields with magnetic vector potential A^a and electric potential ϕ , namely $i\hbar\partial_t\Psi = \hat{H}\Psi$ [with \hat{H} in (3)], follows from that of a non-charged particle in an external mechanical potential V via a minimal-coupling prescription

$$i\hbar\partial_t \rightarrow i\hbar\partial_t - e\phi, \quad -i\hbar D_a \rightarrow -i\hbar D_a - \frac{e}{c}A_a. \quad (15)$$

The stationary equation for the energy eigenvalues E is then

$$\left[\frac{1}{2m}(-i\hbar D_a - \frac{e}{c}A_a)^2 + e\phi + V \right] \psi = E\psi, \quad (16)$$

where $\Psi = e^{-iEt/\hbar}\psi$ (with $\partial_t\psi = 0$). This equation should not depend on the gauge choice of the electromagnetic potentials. The gauge transformation of A_a by a total gradient [6]

$$A_a \rightarrow A_a - D_a\sigma, \quad (17)$$

leaves Eq. (16) invariant if we simultaneously transform ψ as

$$\psi \rightarrow e^{ie\sigma/(c\hbar)}\psi, \quad (18)$$

i.e. by a (local) phase. Transformations (17) and (18) define the electromagnetic abelian $U(1)$ -gauge symmetry. From these remarks we can state the following similarities between the eigenvalue problems (9) and (16), placing the MOTS-QCP analogy in (2) on a sounder structural basis [7]:

i) *Abelian gauge symmetry.* The QCP eigenvalue problem (16) and the MOTS-spectral problem (9) are respectively invariant under transformations (Eqs. (17)-(18) and Lemma 1)

$$\begin{aligned} \text{QCP: } A_a &\rightarrow A_a - D_a\sigma, \quad \psi \rightarrow e^{ie\sigma/(c\hbar)}\psi \\ \text{MOTS: } \Omega_a &\rightarrow \Omega_a - D_a\sigma, \quad \psi \rightarrow e^{-\sigma}\psi. \end{aligned} \quad (19)$$

They both define abelian symmetries of gauge nature: in the QCP case it is the electromagnetic $U(1)$ -gauge transformation

($g'' = g \cdot g'$, with $g = e^{ie\sigma/(c\hbar)}$) relying on the phase invariance of the wave function, whereas for MOTSs it defines a non-compact \mathbb{R}^+ -gauge counterpart ($g'' = g \cdot g'$, with $g = f = e^{-\sigma}$) reflecting the in-built null rescaling (boost) freedom of the MOTS geometric description. In brief, the MOTS-spectral problem presents symmetry transformation properties in full analogy with those of the QCP Schrödinger equation.

ii) *Gauge field potential.* In this symmetry setting, the 1-form Ω_a emerges as the natural gauge field of the \mathbb{R}^+ -gauge group. This endorses, at a structural level, its purely formal correspondence in (2) with the A_a magnetic $U(1)$ -gauge field. We note that the normal connection in (7) admits an interpretation as a gauge connection: $\ell^b D_a^\perp(\psi k_b) = -(D_a - \Omega_a)\psi$.

iii) *Minimal Coupling.* It is at a “dynamical” level where the analogy proves remarkable: the Ω_a field enters in L_S via a standard gauge “minimal coupling” mechanism, namely a shift in the Levi-Civita connection with the gauge connection

$$D_a \rightarrow D_a - \Omega_a. \quad (20)$$

This becomes apparent in the perfect-square version (4) of L_S . Therefore, in full analogy with the minimal coupling mechanism for incorporating the magnetic field in the QCP problem via the shift (15) in the non-charged equations, rotation in a MOTS is switched-on via the minimal coupling (20).

A. MOTSs and a negative “fine structure constant” α

Setting $\hbar=m=c=1$ and introducing a formal complex “fine-structure constant” $\alpha \equiv e^2$, we define the operator family

$$\begin{aligned} L[\sqrt{\alpha}] &= -\frac{1}{2}(D - i\sqrt{\alpha}\Omega)^2 - \frac{\alpha}{4}R_S - \frac{1}{2}G_{ab}k^a\ell^b \\ &= -\frac{1}{2}\Delta + i\sqrt{\alpha}(\Omega \cdot D + \frac{1}{2}D \cdot \Omega) + \frac{\alpha}{2}|\Omega|^2 - \frac{\alpha}{4}R_S - \frac{1}{2}G(k, \ell). \end{aligned} \quad (21)$$

The QCP Hamiltonian corresponds to the (normalized) standard real positive $\alpha = 1$, whereas (half) the MOTS-stability operator corresponds to a negative $\alpha = -1$. Specifically, QCP and MOTS operators are recovered with branch choices: $\hat{H} = L[\sqrt{\alpha} = 1]$ and $L_S/2 = L[\sqrt{\alpha} = -i]$. In this sense, stable MOTSs can be seen as QCPs with negative “fine-structure constant” α . This suggests to import QCP terms to MOTSs.

Terminology for L_S terms. Regarding terms containing the rotation field Ω_a , we refer to $|\Omega|^2$ as the *diamagnetic* term, whereas $\Omega \cdot D$ is the *paramagnetic* term [8]. The divergence $D \cdot \Omega$ is a *gauge-fixing* term and can be chosen by an appropriate transformation (19). For completeness sake, Δ is the *kinematical* term and $G(k, \ell)$ is the *external mechanical* potential. Finally, $R_S/4$ can be referred to as the *electric* potential term. To justify its explicit distinction from the *external mechanical* potential, we consider in the 2-dimensional case the complex scalar \mathcal{K} on \mathcal{S} introduced by Penrose and Rindler [9] as

$$\mathcal{K} = \frac{1}{4}R_S + i\frac{1}{4}\epsilon^{ab}F_{ab}^\Omega, \quad (22)$$

where $F_{ab}^\Omega = D_a\Omega_b - D_b\Omega_a$, namely the curvature of Ω_a . The real and imaginary parts of \mathcal{K} correspond, respectively,

to *electric* and *magnetic* terms. This gravity/electromagnetic analogy in \mathcal{K} has been used to discuss isolated/dynamical horizon source multipoles [10] and to introduce the notions of “vortexes” and “tendexes” in the analysis of dynamical black holes [11]. The present discussion promotes such analogy to a sounder structural level by identifying the symmetry and minimal coupling similarities in the relevant operators.

The ultimate motivation behind this analogy is to explore the transfer of concepts and tools between both problems. This can prove fruitful for the MOTS-spectral problem by profiting of the extensive knowledge accumulated about QCP bound states. In parallel, the development of spinor treatments of MOTS-stability can largely benefit from the presented analogy. In particular, using the Lichnerowicz-Weitzenböck formula to mimic Pauli’s approach to spin can provide insights in the structure of the MOTS second-order operator L_S , whereas Dirac’s approach to spin can open an avenue to a first-order formulation of MOTS-stability (of potential interest in boundary value problems of bulk spinor equations). The GHP formalism [12] can offer additional insights in this setting. We postpone the development of spinor approaches to future studies and focus in the following on the MOTS-spectral problem.

IV. MOTS-SPECTRAL PROBLEM

A. An explicit example: “MOTS-Landau” levels

We consider now the behaviour of the eigenvalues and eigenfunctions in problems (16) and (9), under the complex rotation (from $\sqrt{\alpha} = 1$ to $\sqrt{\alpha} = -i$, in $L[\sqrt{\alpha}]$) that realizes the analogy (2). In this context, and following the spirit of Landau levels of a QCP moving in a constant magnetic field (that provides an explicit example illustrating basic features of such quantum systems), we start by discussing a simple eigenvalue problem for L_S that can be explicitly solved both in the MOTS case and, independently, in the analogous QCP case.

Let us take a 2-sphere $\mathcal{S} = S^2$ with “round” metric $q_{ab} = r^2(d\theta^2 + \sin^2\theta d\varphi^2)$. Decomposing the Hájíček form as $\Omega_a = \epsilon_a^b D_b\omega + D_a\zeta$ with the simplest non-trivial choice $\omega = a \sin\theta, \zeta = 0$ ($a \in \mathbb{R}$), we have: $\Omega = a \sin^2\theta d\varphi$. In vacuum (i.e. $G_{ab} = 8\pi T_{ab} = 0$), the solution to the MOTS-eigenvalue problem for the resulting L_S is given explicitly by

$$\lambda = \frac{1}{r^2} [(\lambda_{\ell m}(a) + 1 - a^2) + i2am], \quad \psi = S_{\ell m}(a, \cos\theta)e^{im\varphi} \quad (23)$$

where $S_{\ell m}(a, \cos\theta)$ are the *prolate* spheroidal functions with eigenvalues $\lambda_{\ell m}(a)$ (21.6.2 in [13]). Standard spherical harmonics are recovered in the limit $a \rightarrow 0$: $\lambda_{\ell m}(a) \rightarrow \ell(\ell+1)$ and $S_{\ell m}(a, \cos\theta) \rightarrow P_{\ell m}(\cos\theta)$. We can now consider the “QCP counterpart” by performing the complex rotation $a \rightarrow ia$ in the operator L_S . The resolution of the new eigenvalue problem leads to eigenvalues $\bar{\lambda}$ and eigenfunctions $\bar{\psi}$

$$\bar{\lambda} = \frac{1}{r^2} (\bar{\lambda}_{\ell m}(a) + 1 + a^2 - 2am), \quad \bar{\psi} = \bar{S}_{\ell m}(a, \cos\theta)e^{im\varphi} \quad (24)$$

where $\bar{S}_{\ell m}(a, \cos\theta) = S_{\ell m}(ia, \cos\theta)$ are now the *oblate* spheroidal functions with eigenvalues $\bar{\lambda}_{\ell m}(a) = \lambda_{\ell m}(ia)$ (cf.

21.6.4 and 21.7.5 in [13]; note $\lambda_{\ell m}(ia) \in \mathbb{R}$). Therefore, at least in this simple example it is verified that eigenvalues λ (eigenfunctions ψ) of the operator $L_S = 2L[-ia]$ [14] can be actually recovered by solving for the “rotated” $a \rightarrow ia$ self-adjoint operator $L[a]$, and then inverting the rotation $a \rightarrow \frac{1}{i}a = -ia$ in the resulting eigenvalues $\bar{\lambda}$ (eigenfunctions $\bar{\psi}$).

B. Analyticity in the “fine structure constant”

The fact that the MOTS-stability operator can be obtained from the QCP Hamiltonian as an analytic continuation of $L[\sqrt{\alpha}]$ ($\sqrt{\alpha} = 1 \rightarrow \sqrt{\alpha} = -i$), together with the discussion of the previous explicit example, raise the following question: *can we recover the MOTS-spectrum ($\alpha = -1$) as an analytic extension of the QCP spectrum ($\alpha = 1$) self-adjoint problem?*

This question dwells naturally in the perturbation theory of linear operators (where $L[\sqrt{\alpha}]$ defines a self-adjoint holomorphic family of type (A) [15]), but giving a fully general answer defines a difficult problem. A given eigenvalue $\lambda(\sqrt{\alpha})$ can be analytically continued along its path in the complex plane, as long as its evolution does not encounter (for the same $\sqrt{\alpha}$) another eigenvalue. But checking this is a hard task even in the explicit example above. On the other hand, our particular setting is free of two potential threats for the analyticity discussion, namely boundaries and function pathologies: *i)* \mathcal{S} has no boundaries (is closed), and *ii)* the functions in $L[\sqrt{\alpha}]$, being induced from the ambient geometry, can be taken as regular as needed. As a third point *iii)*, potential topological issues associated to the underlying $U(1)$ or \mathbb{R}^+ -fibre bundle are absent since such bundle is trivial (we are excluding here the possibility of a non-trivial NUT charge). Supported by these points and in the assumption that the example in IV A contains all the relevant qualitative elements, we propose the following:

Analyticity Conjecture. *Given an orientable closed surface \mathcal{S} and the one-parameter family of operators $L[\sqrt{\alpha}]$ defined in (21), the MOTS-spectrum ($\sqrt{\alpha} = -i$) can be recovered as an “analytic continuation” of the QCP spectrum ($\sqrt{\alpha} = 1$).*

We present this conjecture as an open problem. In case the conjecture proves to be valid [16], the MOTS-stability spectrum problem would be “essentially” reduced to that of the self-adjoint problem of the stationary non-relativistic QCP.

C. Ground state of the charged particle

As a first application, we consider a transfer in the “inverse” sense, by using a MOTS result to calculate the ground state energy E_o of QCPs. In [4] a variational Rayleigh-Ritz-like expression for λ_o is presented. This remarkable result does not follow from the Rayleigh-Ritz characterization, since L_S is generically not selfadjoint. The expression for λ_o is rather obtained by starting from a min-max characterization by Donsker and Varadhan [17], valid for real not necessarily selfadjoint operators. If the conjecture above proves true, the “rotation” $\sqrt{\alpha} \rightarrow -i\sqrt{\alpha}$ in the λ_o of [4] results in a QCP E_o

$$E_o = \inf_{\psi > 0} \int_{\mathcal{S}} (|D\psi|^2 + (e\phi + V + e^2|D\omega_\psi + z|^2) \psi^2) d\mathcal{S} \quad (25)$$

where $A_a = z_a + D_a\zeta$ (with $D_a z^a = 0$), $\int_{\mathcal{S}} \psi^2 \eta_S = 1$ and ω_ψ satisfies, for a given $\psi > 0$, the constraint equation

$$-\Delta\omega_\psi - \frac{2}{\psi} D_a \psi D^a \omega_\psi = \frac{2}{\psi} z^a D_a \psi. \quad (26)$$

This expression for E_o , “blindly” transported from the MOTS result, has two virtues as compared with the straightforward evaluation of the Rayleigh-Ritz $E_o = \inf_{\|\psi\|=1} \int_{\mathcal{S}} \psi^* \hat{H} \psi \eta_S$: i)

it is explicitly gauge-invariant, since the term $D_a A^a (= \Delta\zeta)$ is absent; and ii) the paramagnetic term is recast as a diamagnetic one, something of potential interest in numerical strategies. Then, a crucial point is that the “blind” expression (25) for QCPs can actually be proved: starting from the (now valid) Rayleigh-Ritz expression and adapting the steps in [4] to the MOTS/QCPs analogy, expression (25) follows. This is remarkable since, although it is known [17] that Rayleigh-Ritz and Donsker-Varadhan expressions coincide when they both apply (namely, selfadjoint real operators), they cannot be generically reduced to one another (in particular in our setting $A_a \neq 0$). Therefore, the fact that (25) still holds in the fully general case offers a first non-trivial test of the conjecture.

D. Semi-classical approach to the MOTS spectrum

As a second example, the effective reduction to a selfadjoint problem would open a particular avenue to the use of approximate tools in the MOTS spectral problem, by transferring the semi-classical tools for QCPs [18]. More specifically, starting from the MOTS-stability operator, we can in a first step consider the analogous QCP Hamiltonian $\hat{H}[\sqrt{\alpha} \in \mathbb{R}]$, and in a second step its corresponding classical Hamiltonian function

$$H_{cl}[\sqrt{\alpha}](x, p) = (p - \sqrt{\alpha}\Omega)^2 + \frac{1}{2}R_S - G(k, \ell), \quad (27)$$

defined on the cotangent bundle $T^*\mathcal{S}$ by reverting the “quantization rule”, $p_i \rightarrow -iD_i$. Approximate eigenvalues and eigenfunctions for the MOTS problem could then be obtained, assuming the validity of the conjecture, by applying semi-classical tools on $H_{cl}[\sqrt{\alpha}]$ and then evaluating the explicit result on $\sqrt{\alpha} \rightarrow -i$. Whereas WKB techniques could be appropriate in separable problems, generic cases (notably, $\Omega_a \neq 0$) would need to resort to the rich semi-classical tools developed in the setting of quantum chaos studies (e.g. [19, 20]).

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- [1] Jona-Lasinio, G., Prog. Theor. Phys. Suppl. **184**, 1–15 (2010).
 - [2] Andersson, L., Mars, M., Simon, W., Phys. Rev. Lett. **95**, 111,102 (2005)
 - [3] Jaramillo, J.L., Phys. Rev. **D89**, 021,502 (2014)
 - [4] Andersson, L., Mars, M., Simon, W., Adv. Theor. Math. Phys. **12**, 853–888 (2008)
 - [5] Mars, M., Class.Quant.Grav. **29**, 145,019 (2012).
 - [6] The electric potential ϕ stays invariant $\phi \rightarrow \phi - \frac{1}{c}\partial_t\sigma$ under gauge transformations compatible with stationarity, $\partial_t\sigma = 0$.
 - [7] A further point could be the understanding of MOTS-stability, $\lambda_o \geq 0$, as a MOTS-counterpart of a positivity condition on the quantum ground state E_o , refining quantum stability. This is however delicate, since the operator correspondance (2) does not necessarily preserve eigenvalue signs (see section IV).
 - [8] Galindo, A., Pascual, P., Quantum Mechanics II. Springer (1991)
 - [9] Penrose, R., Rindler, W., Spinors and space-time. Volume 2. Cambridge University Press (1986)
 - [10] Ashtekar, A., Engle, J., Pawłowski, T., Van Den Broeck, C., Class. Quant. Grav. **21**, 2549 (2004)
 - [11] Owen, R. et al., Phys. Rev. Lett. **106**(15), 151,101 (2011).
 - [12] Geroch, R., Held, A., Penrose, R., J. Math. Phys. **14**, 874–881 (1973).
 - [13] Abramowitz, M., Stegun, I.A., Handbook of Mathematical Functions. Dover Publications, New York (1964)
 - [14] $L[-ia]$ assumes implicitly a Hájiček form $\Omega = \sin^2\theta d\varphi$.
 - [15] Kato, T., Perturbation theory for linear operators. Reprint of the corr. print. of the 2nd ed. 1980., Berlin: Springer-Verlag (1995)
 - [16] Even without a continuation argument, the prescription $\sqrt{\alpha} \rightarrow -i\sqrt{\alpha}$ to recover MOTS spectra from QCPs could still hold.
 - [17] Donsker, M.D., Varadhan, S.R.S., Proc. Nat. Acad. Sci. USA **72**, 780–783 (1975)
 - [18] But note that “non-selfadjoint” semi-classical tools also exist.
 - [19] Berry, M.V., Semiclassical Mechanics of regular and irregular motion, in Les Houches Lecture Series Session XXXVI, North Holland, Amsterdam, 171-271. (1983)
 - [20] Nonnenmacher, S., Séminaire Poincaré XIV, 177–220 (2010).